Cluster dynamics: a primer

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[*Warning: These notes focus on deriving appropriate scalings; constant factors of order unity are not traced in a self-consistent manner.*]

We will be dealing with an N-body non-relativistic gravitationally interacting system ($N \gg 1$), which we will refer to as a *cluster*. For now, assume a single mass species with component mass m, so the total cluster mass is $M = Nm$. Further assume a spherically symmetric cluster of size R.[1] The number density is then

$$n \sim \frac{N}{R^3} \ , \tag{1}$$

the typical velocity dispersion is

$$v_{\text{disp}} \sim \sqrt{\frac{GM}{R}} = \sqrt{\frac{GNm}{R}} \tag{2}$$

and the escape velocity is a few times v_{disp}.

Consider a binary with orbital separation (semi-major axis) a, and corresponding orbital velocity

$$v_{\text{orb}} \sim \sqrt{\frac{Gm}{a}} \ . \tag{3}$$

This binary will gravitationally interact with single stars flying by with typical velocity v_{disp}. If the orbital energy $-Gm^2/(2a)$ is smaller in magnitude than the kinetic energy of

[1]There are different definitions for the size (half-mass radius, half-light radius, R_{200}, etc.) and different radial density profiles (uniform, Spitzer, isothermal, Plummer, King, NFW, etc.). We will ignore these complications for now.

the interacting object $mv_{\rm disp}^2/2$, i.e., $v_{\rm orb} \lesssim v_{\rm disp}$, the so-called *soft* binary is likely to be disrupted by the interaction. If, on the other hand, the binary is *hard* and $v_{\rm orb} > v_{\rm disp}$, then $2+1$ dynamical interactions will further harden (tighten) the binary (Heggie 1975). Thus, hard binaries harden while soft binaries are destroyed, with the boundary falling at

$$a_{\rm hard} \sim \frac{Gm}{v_{\rm disp}^2} \sim \frac{R}{N} \ . \tag{4}$$

On average, interactions with stars whose total mass is a few times the mass of the binary are necessary to harden the binary by one e-folding of semimajor axis (Quinlan 1996). Typically, the lightest of the three interacting objects will be ejected from the binary; thus, if the interloper is heavier than either of the binary components, it is likely to substitute in. Such interactions will also cause the binary to sample a thermal eccentricity distribution, $p(e) = 2e$.[2]

Even in the absence of primordial binaries, binaries will generically form through three-body dynamical interactions (a third body is necessary to carry away the excess energy in order to create a bound system). In order to form a hard binary, it is necessary to bring three stars to a distance $\lesssim a_{\rm hard}$ from each other. There are $\sim N^3$ distinct volumes of radius $a_{\rm hard}$ in the whole cluster of size R. The probability of finding three of N objects within any of these at a given time is $\approx C_3^N \left(N^{-3}\right)^3 \approx N^{-6}/6$, and the probability that at least one of the small volumes will have 3 objects is $\approx N^3 \times N^{-6}/6 \sim N^{-3}$. The timescale for the objects to be re-arranged between volumes, i.e., the timescale for an object to cross a given volume while traveling at $v_{\rm disp}$, is $\sim (R/N)/v_{\rm disp}$. Therefore, the timescale for a binary to form is

$$\tau_{\rm bin,form} \sim \frac{R}{N v_{\rm disp}} N^3 = \frac{N^2 R}{v_{\rm disp}} \sim \frac{N^2 R^{3/2}}{(GNm)^{1/2}} \sim N^2 \tau_{\rm cross} \ , \tag{5}$$

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[2]In practice, the neither the energy distribution nor the eccentricity distribution ever reach the thermal distribution (Geller et al. 2019).

where $\tau_{\rm cross} \sim R^{3/2}\,(GM)^{-1/2}$ is the cluster crossing timescale.

The rate at which interlopers will strongly interact with a given binary, i.e., pass by within a distance a of the binary, is $\Gamma = n\sigma v_{\rm disp}$, where σ is the interaction cross-section. For soft binaries, the interaction cross-section is just the geometrical cross-section, $\sigma \sim \pi a^2$. However, when the binary is hard, the relatively slowly moving interlopers experience gravitational focusing. Consider the extreme case $v_{\rm orb} \gg v_{\rm disp}$, which allows us to treat the binary as a point particle of mass $2m$. If the interloper approaches the binary from infinity with impact parameter b, it has an initial angular momentum $mv_{\rm disp}b$. If the periapsis distance is a, the velocity at periapsis is very nearly $\sqrt{4Gm/a}$ and the angular momentum there is $m\sqrt{4Gma}$. Thus, conservation of angular momentum dictates that $b \sim 2\sqrt{Gma}/v_{\rm disp}$, and the cross-section for interlopers to get within a distance a of the binary is $\pi b^2 \sim 4\pi Gma/v_{\rm disp}^2$. Note that the cross-section scales linearly rather than quadratically with a once gravitational focusing is included. The interaction timescale is then

$$\tau_{\rm int} = \Gamma^{-1} \sim \frac{1}{n\sigma v_{\rm disp}} \sim \frac{v_{\rm disp}}{nGma}. \tag{6}$$

For equal-mass binaries and interlopers of the same mass, only $O(1)$ interactions are needed to harden the binary by a factor of ~ 2. Because the last e-folding in hardening the binary takes the longest time, $\tau_{\rm int}$ is a reasonable order-of-magnitude approximation for both the time to the next interaction and for the time it has taken the binary to harden to the current orbital separation through three-body $2+1$ interactions.

Because each interaction carries away a significant fraction of the binary's orbital energy, the interloper is kicked with a velocity $\sim v_{\rm orb}$. Conservation of linear momentum for the binary–interloper system therefore implies that the binary must get a recoil kick with a velocity $\sim v_{\rm orb}/2$. The escape velocity for a globular cluster is only a factor of a few greater than the velocity dispersion (e.g., if $v_{\rm disp} = 10$ km/s, the escape velocity may be $\lesssim 50$ km/s). Thus, recoil kicks will eject the binary once its orbital velocity reaches $v_{\rm orb} \approx 10 v_{\rm disp}$. Since $v_{\rm orb} \sim v_{\rm disp}$ at the hard-soft binary, and $v_{\rm orb} \propto a^{-1/2}$, the binary can reach a minimum semimajor axis $a_{\rm eject}$ approximately two orders of magnitude smaller than $a_{\rm hard}$ before being ejected. Binaries tighter than

$$a_{\rm eject} \sim 0.01 a_{\rm hard} \tag{7}$$

can only remain in the cluster if gravitational-wave hardening takes over as the dominant forcing mechanism before the binary reaches this orbital separation and can be ejected.

Thus, the fate of binaries is determined by a comparison of $\tau_{\rm int}$, the Hubble time $\tau_{\rm H} = 14$ Gyr, and the gravitational-wave merger timescale $\tau_{\rm GW}$ (Peters 1964):

$$\begin{aligned} \tau_{\rm GW}(e=0) &= 1.6 \text{ Gyr} \left(\frac{a}{0.01 \text{ AU}}\right)^4 \left(\frac{m}{M_\odot}\right)^{-3} \\ \tau_{\rm GW}(e\to 1) &= 32 \text{ Gyr} \left(\frac{a}{0.01 \text{ AU}}\right)^4 \left(\frac{m}{M_\odot}\right)^{-3} (1-e)^{7/2} \,. \end{aligned} \tag{8}$$

Several cases are possible:

- If the total 2+1 hardening and GW emission timescale $\tau_{\rm int} + \tau_{\rm GW}(e=0) < \tau_{\rm H}$ at some a between $a_{\rm hard}$ and $a_{\rm eject}$, the binary will merge inside the cluster through a sequence of $2+1$ hardening interactions and gravitational-wave emission.

- Otherwise, if $\tau_{\rm int} + \tau_{\rm GW}(e=0) \geq \tau_{\rm H}$ for all $a \in [a_{\rm eject}, a_{\rm hard}]$, but $\tau_{\rm int} < \tau_{\rm H}$ at $a_{\rm eject}$, the binary may either merge inside the cluster if $2+1$ interactions happen to drive it to a sufficiently high eccentricity to reduce $\tau_{\rm GW}$ so that $\tau_{\rm int} + \tau_{\rm GW}(e) < \tau_{\rm H}$, or it may be ejected, and may or may not subsequently evolve outside the cluster depending on its $\tau_{\rm GW}$ at ejection.

- If neither of these holds, i.e., if $\tau_{\rm int} + \tau_{\rm GW}(e=0) \geq \tau_{\rm H}$ for all $a \in [a_{\rm eject}, a_{\rm hard}]$ *and* $\tau_{\rm int} > \tau_{\rm H}$ at $a_{\rm eject}$ the binary will remain in the cluster and stall at the orbital separation at which $\tau_{\rm int}$ exceeds $\tau_{\rm H}$.

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A couple more timescales are worth mentioning. The relaxation timescale is the time for the cluster to thermalise, i.e., for a typical star to change its velocity by order of its velocity. That can be achieved by a single strong encounter with an interloper approaching within a distance $a_{\rm hard}$. This is just $\tau_{\rm int}(a_{\rm hard}) \sim v_{\rm disp}^3 G^{-2} m^{-2} n^{-1}$. It turns out that relaxation is more efficiently driven by many weak scatterings rather than a few strong ones, which give rise to a so-called Coulomb logarithm; the relaxation time is a factor of $\sim \log N$ lower than $\tau_{\rm int}(a_{\rm hard})$, or $\sim N\tau_{\rm cross}/\log N$ using $n \sim N/R^3$.

The evaporation timescale (the time for a significant fraction of the objects in the cluster to be ejected) is ~ 100 times longer than the relaxation timescale, because $< 1\%$ of stars with a Maxwellian velocity distribution centred on $v_{\rm disp}$ will exceed the escape velocity and evaporate from the cluster, and a relaxation time is required to repopulate this high-velocity tail of the stellar phase space distribution.

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