When will the core of an ultra-relativistic jet be visible to an off-axis observer?

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Suppose the core of the jet is initially moving with an ultra-relativistic Lorentz factor $\Gamma_0 \equiv (1-\beta_0^2)^{-1/2} \equiv (1-(v/c)^2)^{-1/2} \gg 1$. It propagates, without sideways expansion, through the interstellar medium assumed to have uniform density $\rho \approx n m_p$, where m_p is the proton mass. Because of relativistic beaming, any synchrotron radiation would be confined to an angle $1/\Gamma$; the jet core would therefore only be visible to an off-axis observer located at angle θ when Γ drops to $\Gamma = 1/\theta$ as a result of the jet's interaction with the interstellar medium.

If the jet carries mass M_0 and sweeps up mass fM_0, energy and momentum conservation dictate

$$\Gamma_0 + f = (1+f)\gamma\Gamma; \tag{1}$$

$$\Gamma_0\beta_0 = (1+f)\gamma\Gamma\beta, \tag{2}$$

where γ is the internal Lorentz factor within the jet. Solving these (see, e.g., Paczynski & Rhoads, 1993 – we use their notation) yields

$$\beta = \frac{\Gamma_0\beta_0}{\Gamma_0+f}; \tag{3}$$

$$\Gamma = \frac{\Gamma_0+f}{\sqrt{1+2\Gamma_0 f+f^2}}. \tag{4}$$

Consider radiation emitted when the jet has reached a distance R from the initial explosion. This radiation would arrive at an on-axis observer later than information propagating at the speed of light from the initial explosion (say, gravitational waves) by a time

$$\Delta t(\theta=0) = \int_0^R \frac{dr}{\beta c} - \frac{R}{c}. \tag{5}$$

If the jet core has subtends a solid angle $d\Omega$, then $f = (d\Omega/3)r^3\rho/M_0$, where r is the instantaneous distance of the jet from the explosion site. Hence,

$$\begin{aligned}\Delta t(\theta=0) &= \int_0^R \frac{\Gamma_0+f}{c\Gamma_0\beta_0}dr - \frac{R}{c} = \int_0^R \frac{\Gamma_0 + d\Omega/3r^3\rho/M_0}{c\Gamma_0\beta_0}dr - \frac{R}{c} = \frac{R}{c\beta_0} - \frac{R}{c} + \frac{Rf(R)}{4c\Gamma_0\beta_0} \\ &= \frac{R}{c\Gamma_0\beta_0}[\Gamma_0 + f(R)/4 - \Gamma_0\beta_0] = \frac{R}{2c\Gamma_0^2\beta_0}[1+\frac{f(R)\Gamma_0}{2}],\end{aligned} \tag{6}$$

where we used $\Gamma - \Gamma\beta = 1/(2\Gamma)$ in the last simplification.

Now, specifically consider the regime when the mass of swept up material is such that $\Gamma_0 \gg f \gg 1$. In this case, $\Gamma \approx \sqrt{\Gamma_0}/\sqrt{2f}$ (note this also corresponds to $1 \ll \Gamma \ll \Gamma_0$), and hence $f \approx \Gamma_0/(2\Gamma^2)$. Then we can further simplify

$$\Delta t(\theta=0) \approx \frac{R}{2c\Gamma_0^2\beta_0}[1+\frac{\Gamma_0^2}{4\Gamma^2}] \approx \frac{R}{8c\Gamma^2}. \tag{7}$$

If the observer is located off-axis, there is an extra geometric factor in the delay time,

$$\Delta t(\theta) = \int_0^R \frac{dr}{\beta c} - \frac{R}{c}\cos\theta = \Delta t(\theta=0) + \frac{R}{c}(1-\cos\theta) \approx \frac{R}{8c\Gamma^2} + \frac{R}{c}\frac{\theta^2}{2} \approx \frac{5R\theta^2}{8c}, \tag{8}$$

where we assumed $\theta = 1/\Gamma \ll 1$.

This expression can be expressed in terms of the initial jet kinetic energy $E_K = M_0(\Gamma_0-1)c^2 \approx M_0\Gamma_0 c^2$ and the initial jet core opening angle θ_0, corresponding to $d\Omega = 2\pi\theta_0^2$ for a two-sided jet. Equating $f(R) = (d\Omega/3)R^3 n m_p/M_0$ and $f \approx \Gamma_0/(2\Gamma^2) \approx \Gamma_0\theta^2/2$, we can write

$$R \approx \left(\frac{3M_0\Gamma_0\theta^2}{4\pi\theta_0^2 n m_p}\right)^{1/3} = \left(\frac{3}{4\pi m_p c^2}\frac{E_K}{n}\right)^{1/3}\left(\frac{\theta}{\theta_0}\right)^{2/3}. \tag{9}$$

Hence

$$\Delta t(\theta) \approx \frac{5}{8c}\left(\frac{3}{4\pi m_p c^2}\frac{E_K}{n}\right)^{1/3}\left(\frac{\theta}{\theta_0}\right)^{2/3}\theta^2 \approx 160\left(\frac{E_K}{10^{50}\,\mathrm{erg}}\right)^{1/3}\left(\frac{n}{10^{-4}\,\mathrm{cm}^{-3}}\right)^{-1/3}\left(\frac{\theta}{\theta_0}\right)^{2/3}\left(\frac{\theta}{20^\circ}\right)^2 \text{ days.} \tag{10}$$

This expression can be further simplified through the isotropic equivalent energy of the jet core, $E_0 = 2E_K/\theta_0^2$:

$$\Delta t(\theta) \approx 420 \left(\frac{E_0}{3\times 10^{52}\,\mathrm{erg}}\right)^{1/3} \left(\frac{n}{10^{-4}\,\mathrm{cm}^{-3}}\right)^{-1/3} \left(\frac{\theta}{20^\circ}\right)^{8/3} \mathrm{days}. \tag{11}$$

If the jet does expand sideways after the jet break (i.e., once $1/\Gamma > \theta_0$), it would reach an opening angle of $1/\Gamma$ by the time the light was emitted to an observer at $\theta = 1/\Gamma > \theta_0$. Then there would be no additional geometric factor to consider. Furthermore, sideways expansion would increase the amount of swept-up material: $d\Omega(r) = 2\pi\theta(r)^2 = 2\pi/\Gamma(r)^2$ for a two-sided jet post-break. This more complicated behaviour of $f(r)$, with $df/dr = d\Omega r^2\rho/M_0$ after the jet break, leads to a more challenging integral for Δt. However, it's sufficient to note that $f(r)$ will grow very rapidly after jet break: using $f \approx \Gamma_0/(2\Gamma^2)$ yields the differential equal

$$\frac{df}{dr} = \frac{d\Omega r^2\rho}{M_0} = \frac{2\pi r^2\rho}{\Gamma^2 M_0} = \frac{4\pi\rho}{\Gamma_0 M_0} r^2 f. \tag{12}$$

The solution is

$$f \propto \exp\left(\frac{4\pi}{3}\frac{\rho}{\Gamma_0} M_0 r^3\right). \tag{13}$$

The normalization is set by $f(R_{\rm break}) = \Gamma_0\theta_0^2/2$ at the jet break radius $R_{\rm break} = [3M_0\Gamma_0/(4\pi\rho)]^{1/3}$, which follows from equation (9):

$$f(r) = \frac{\Gamma_0\theta_0^2}{2e} \exp\left[(r/R_{\rm break})^3\right]. \tag{14}$$

The jet will spread to $\Gamma = 1/\theta$ and reach $f(R) = \Gamma_0\theta^2/2$ at a distance $R = \left(2\log(\theta/\theta_0)\right)^{1/3} R_{\rm break}$.

We can now compute the contribution to the delay time Δt from the time after the jet break

$$\Delta t_{>\rm break} \approx \int_{R_{\rm break}}^{R} \frac{f}{c\Gamma_0\beta_0} dr \approx \frac{\theta_0^2 R_{\rm break}}{2e\,c} \int_1^{R/R_{\rm break}} \exp\left[x^3\right] dx. \tag{15}$$

We can write down the integral with the aid of an incomplete gamma function; however, it is sufficient for our purposes to note that the integral is very well approximated by

$$\int_1^y \exp\left(x^3\right) dx \approx \frac{\exp\left(y^3\right)}{3y^2} \tag{16}$$

as long as $y > 1.3$. Thus, as long as $\theta \gtrsim 3\theta_0$, the post-break contribution to the delay time can be approximated as

$$\Delta t_{>\rm break} \approx \frac{\theta_0^2 R_{\rm break}}{6e\,c} \left(2\log(\theta/\theta_0)\right)^{-2/3} \left(\frac{\theta}{\theta_0}\right)^2. \tag{17}$$

The contribution of the time before the jet break to the delay time is at least a factor of a few smaller than this as long as $\theta \gtrsim 3\theta_0$:

$$\Delta t_{<\rm break} = \frac{R_{\rm break}\theta_0^2}{8c}. \tag{18}$$

The total delay time for an expanding jet in this regime is therefore

$$\Delta t = \Delta t_{<\rm break} + \Delta t_{>\rm break} \approx \frac{\theta_0^2 R_{\rm break}}{c}\left[\frac{1}{8} + \frac{1}{6e}\left(2\log(\theta/\theta_0)\right)^{-2/3}\left(\frac{\theta}{\theta_0}\right)^2\right] \approx \left(\frac{3M_0\Gamma_0}{4\pi n m_p}\right)^{1/3} \left(2\log(\theta/\theta_0)\right)^{-2/3} \frac{\theta^2}{6e\,c}. \tag{19}$$

Approximating $\left(2\log(\theta/\theta_0)\right)^{-2/3} \sim 0.5$ (good for a broad range of $\theta \sim$ few $\times\,\theta_0$) and again expressing the result in terms of the jet energy E_K, we conclude that

$$\Delta t_{\rm expand} \approx 8 \left(\frac{E_K}{10^{50}\,\mathrm{erg}}\right)^{1/3} \left(\frac{n}{10^{-4}\,\mathrm{cm}^{-3}}\right)^{-1/3} \left(\frac{\theta}{20^\circ}\right)^2 \mathrm{days}. \tag{20}$$

Note that the numerical pre-factor differs from that given in Granot et al. (2017). See Sari, Piran & Halpern (1999) for a discussion of the different assumptions between that can lead differences of up to factors of 20 in the estimated timescales.