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Radial diffusion timescale in a spherical medium with a radius-dependent mean free path

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ABSTRACT

We compute the time for the radial diffusion of a photon from the origin to some radius R (say, the photosphere) in a spherically symmetric material with a mean free path $\langle\ell\rangle(r)$ that depends on the radial coordinate.

1. INTRODUCTION

While the problem of diffusion, especially in spherical symmetry, is regularly addressed in astrophysics, analytical treatments typically consider a simplified situation with a constant mean free path $\langle\ell\rangle \equiv (\rho\kappa)^{-1}$, where ρ is the density and κ is the opacity. This problem has a well known solution. The optical depth of the origin is $\tau = R/\langle\ell\rangle$, assuming the mean free path becomes infinite beyond radius R. The diffusion time is then $T_{\rm diff} = R\tau/(2c) = R^2/(2c\langle\ell\rangle)$ (e.g., Chandrasekhar 1943).

However, we have been unable to find a correct analytical solution to the more physically relevant problem of diffusion in spherical symmetry with a radial dependence of the mean free path. We define this problem as follows: the photon starts at the origin, $r = 0$; the problem is spherically symmetric; $\langle\ell\rangle(r)$ is known; the probability of scattering while traversing a path segment $dr \ll \langle\ell\rangle$ is $dr/\langle\ell\rangle$; each scattering is isotropic; the photon travels at speed c between scatterings; we are looking for the mean time required for a photon to first reach radius $r = R$.

The only solution we did find, by Mitalas & Sills (1992), considered precisely this problem when computing the timescale for photon diffusion from the centre of the Sun to the surface – a typical situation in which both density and opacity, and thus the mean free path, vary with radius. Unfortunately, their solution was incorrect.

Of course, the problem as defined above is highly idealised and we are not aware of any astrophysical context in which it is strictly relevant. In particular, in any real astrophysical environment, photons will be absorbed and re-emitted as well as scattered, making it meaningless to talk about the diffusion time of a photon; in fact, the very photon number is not conserved.

Nonetheless, the calculation provides a nice illustration of both the application of a diffusion equation (section 2) and of a Monte Carlo numerical simulation (section 3). In fact, some of the authors struggled for longer than they care to admit with getting to analytical and numerical approaches to agree, and we highlight our errors below, which may be of interest to some readers.

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2. ANALYTICAL SOLUTION

The radial diffusion equation for the photon density n in 3 dimensions is (e.g., Rybicki & Lightman 1986)

$$\dot{n} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\langle\ell\rangle c}{3}\frac{dn}{dr}\right), \tag{1}$$

where $\dot{n}$ denotes the time derivative of n.

In a steady state in which photons are produced at the origin at the rate $\dot{N}$ before diffusing outward, n is constant everywhere: $\dot{n} = 0$. This implies that $r^2\langle\ell\rangle dn/dr = -k$, where k does not depend on radius. Then

$$\frac{dn}{dr} = -\frac{k}{r^2\langle\ell\rangle}. \tag{2}$$

The rate of flow of photons moving through any radius $0 < r \leq R$ is the same, $\dot{N}$, which is related to the normalising constant k by

$$\dot{N} = \frac{4\pi kc}{3}, \tag{3}$$

where we the geometric factor of $1/3$ is strictly accurate only when the material is optically thick.

The photon density as a function of radius n can be obtained by integrating dn/dr inward from R. If R is at the photosphere, it may be tempting to set the boundary condition there to be $n(R) = 0$, but this is a mistake, albeit leading to only a small correction in most cases of interest. In fact, the boundary condition at the point from which photons freely stream outward is $n(R) = \dot{N}/(4\pi R^2(c/3)) = k/R^2$, so the photon density at $r \leq R$ is given by

$$n(r) = \int_R^r \frac{-k}{r'^2\langle\ell\rangle(r')}dr' + \frac{k}{R^2}. \tag{4}$$

We can obtain the total number of photons N in the spherical domain $r \leq R$ by integrating $n(r)$ over the sphere:

$$N = 4\pi \int_0^R n(r)r^2 dr. \tag{5}$$

Finally, the diffusion timescale is given by the number of photons in the domain divided by their escape rate,

$$\begin{aligned} T_{\rm diff} &= \frac{N}{\dot{N}} = \frac{3\int_0^R n(r)r^2dr}{kc} \\ &= \frac{3}{c}\int_0^R r^2\left(\int_r^R \frac{dr'}{r'^2\langle\ell\rangle(r')}\right)dr + \frac{R}{c}, \end{aligned} \tag{6}$$

where the arbitrary normalisation has dropped out, as expected. Eq. (6) is the desired analytical diffusion timescale.

Note that this calculation used the Eddington approximation to obtain a geometric factor of 1/3 for the radial photon flux. This breaks down near the photosphere, so the last term R/c in Eq. (6) is approximate. However, this term is generally subdominant by a factor of order the optical depth to the center of the star.

For example, for a radially constant mean free path $\langle\ell\rangle = \ell_0$, Eq. (6) becomes

$$T_0 = \frac{3}{c\ell_0}\int_0^R r^2\left(\frac{1}{r} - \frac{1}{R}\right)dr + \frac{R}{c} = \frac{R^2}{2c\ell_0} + \frac{R}{c}, \tag{7}$$

which matches our expectations other than the (approximate) boundary term, which is suppressed by a factor of R/ℓ_0.

Some readers may be used to the diffusion time being written as $T_{\rm diff} = R\tau/c$, where the optical depth is $\tau = R/\ell_0$, yielding $T_{\rm diff} = R^2/(c\ell_0)$. However, if the mean free path is constant, the actual free path ℓ between scatterings follows an exponential distribution, $p(\ell) = 1/\ell_0 \exp(-l/\ell_0)$. Then $\langle\ell^2\rangle = 2\ell_0^2 = 2\langle\ell\rangle^2$; the actual spreading out of a random walk distribution, which is determined by $\langle\ell^2\rangle$, is a factor of 2 larger than the naive guess, and the diffusion time is a factor of two smaller.

3. MONTE CARLO CONFIRMATION

We can confirm that these timescales are correct with numerical experiments. We have tried several of these, including $\langle\ell\rangle = R/100 * (r/R)^a$ for suitable choices of a such as $a = 0.5$, starting slightly offset from $r = 0$ to avoid being stuck there; or $\langle\ell\rangle = \ell_0 \exp(r/R)$. We report the results for the latter experiment below.

We consider $\langle\ell\rangle(r) = \ell_0 \exp(r/R)$. In the code below, $\ell_0 = 1$ (this sets the code units) and $R = 100$.

```
W=10000; %number of Monte Carlo iterations
R=100; %location of photosphere
l0=1; %mean free path scale length
dr=0.01; %micro step for integrating along free path
nsteps=zeros(W,1); %number of steps tracker
pathlength=zeros(W,1); %integrated path length tracker
for(i=1:W),
    r=[0 0 0]; %current particle location
    d=0; %distance from origin
    while(d<R),
        phi=2*pi*rand(); %isotropic azimuthal angle
        sintheta=-1+2*rand(); %isotropic polar angle
        costheta=sqrt(1-sintheta^2);
        direction = [costheta*cos(phi), ...
         costheta*sin(phi), sintheta];
        step=0; pcontinue=1;
        %evaluate scatter probability over a ray
        %decomposed into micro-steps
        %using the local mean free path
        while(rand<pcontinue),
            step=step+dr;
            r=r+direction*dr;
            d=sqrt(r(1)^2+r(2)^2+r(3)^2);
            l=l0*exp(d/R); %local mean free path
            %probability of scattering is dr/l
            pcontinue=1-dr/l;
        end;
        nsteps(i)=nsteps(i)+1;
        pathlength(i)=pathlength(i)+step;
    end;
end;
```

Note that care must be taken to integrate along the path between scattering interactions in order to determine the free path. The scattering probability is $dr/\langle\ell\rangle(r)$ for $dr \ll \langle\ell\rangle(r)$. Sampling the path from the exponential distribution $p(\ell) = 1/\langle\ell\rangle \exp(-l/\langle\ell\rangle)$ is incorrect unless $\langle\ell\rangle$ is constant.

This yields the distribution of path lengths for escaping photons shown in Figure 1 (we also show the total number of scatters, but note that the free paths between these scatters do not have a constant size).

The mean diffusion time (in units of $c = 1$) is $T_{\rm diff} = 2730 \pm 15$ where the uncertainty on the mean diffusion time is estimated as the standard deviation of the diffusion times divided by the square root of the number of samples ($W = 10,000$ in this case). Integrating Equation (6) yields $T_{\rm diff} = 2742$. The analytical computation matches the numerical result.

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REFERENCES

Chandrasekhar, S. 1943, Reviews of Modern Physics, 15, 1

Mitalas, R., & Sills, K. R. 1992, ApJ, 401, 759

Rybicki, G. B., & Lightman, A. P. 1986, Radiative Processes in Astrophysics

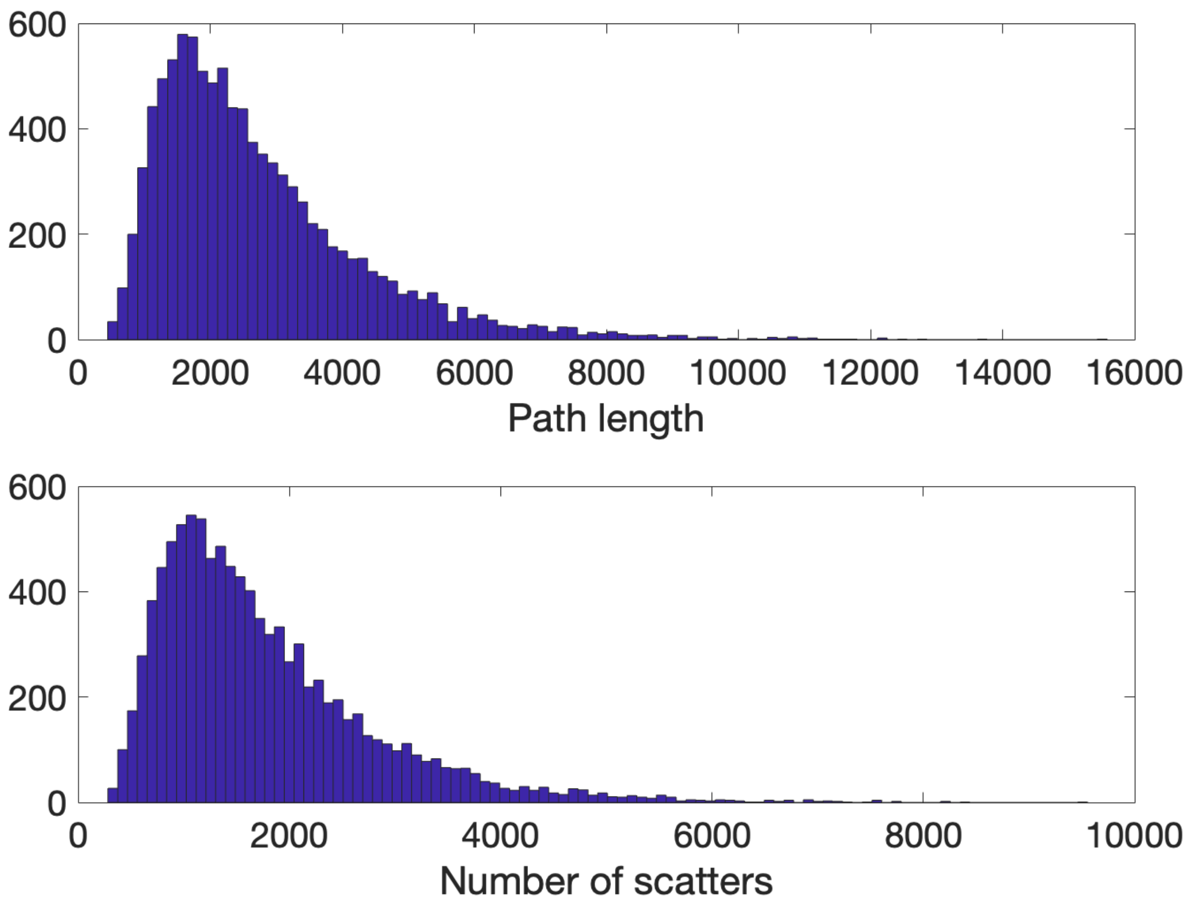

Figure 1. Histograms of path lengths (top) and number of scatters (bottom) for photons escaping from the origin of a sphere of radius $R = 100$ with mean free path $\langle \ell \rangle(r) = \exp(r/R)$.