Common Envelope Timescales

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Consider a donor of mass M and radius R and an inspiralling companion of mass m and radius r (we will assume $m \ll M$ for simplicity, though this is not strictly necessarily and will only lead to order unity errors when $m \sim M$, and of course $r \ll R$). The orbital energy is $E \sim GMm/R$. The donor's average density (we will ignore factors of order unity throughout) is $\langle\rho_{\rm d}\rangle \equiv M/R^3$; the companion's average density is $\langle\rho_{\rm c}\rangle \equiv m/r^3$; and ρ is the density at the current location of the companion.

The companion's Keplerian orbital velocity is $v = \sqrt{GM/R}$ and the dynamical timescale (orbital period) is $\tau_{dyn} = R/v$. The Bondi-Hoyle radius of the companion is $r_B \sim Gm/v^2 = R(m/M)$ (we are assuming super-sonic motion here, $v > c_s$, where c_s is the speed of sound). If $r_B > r$, i.e., $R > (M/m)r$ (this is the case, e.g., for compact-object companions), Bondi-Hoyle drag dominates, and the effective cross-section is $\sigma \sim r_B^2 \sim R^2(m/M)^2$. On the other hand, if $r > r_B$, ram-pressure drag dominates, and the effective cross-section is $\sigma \sim r^2$.

The mass-accretion rate is $\dot{m} = C_A \rho v \sigma$, where C_A is the dimensionless accretion coefficient. The drag force is $F = C_D \rho v^2 \sigma$, where C_D is the dimensionless drag coefficient. The energy dissipation rate is then $\dot{E} = -C_D \rho v^3 \sigma$. Numerical experiments show that, unlike C_A, the drag coefficient C_D is almost always of order unity, so we will generally ignore it below.

The inspiral timescale is $\tau_{insp} \equiv E/|\dot{E}|$. Thus,

$$\frac{\tau_{insp}}{\tau_{dyn}} \sim \frac{GMm}{R^2 \rho v^2 \sigma} = \frac{m}{R\rho\sigma}. \tag{1}$$

In the Bondi-Hoyle regime, $\sigma \sim R^2(m/M)^2$, so

$$\frac{\tau_{insp}}{\tau_{dyn}} \sim \frac{M^2}{R^3 m \rho} = \frac{M}{m}\frac{\langle\rho_{\rm d}\rangle}{\rho}. \tag{2}$$

As an example, consider the inspiral of a neutron star into the convective envelope of a red supergiant en route to forming a double neutron star system. In this case, the envelope's density is not too far off from uniform, so $\langle\rho_{\rm d}\rangle/\rho$ is perhaps only a few, as is M/m – so the inspiral will proceed over a few orbits, which matches the findings of numerical simulations.

In the ram-pressure regime, $\sigma \sim r^2$, so

$$\frac{\tau_{insp}}{\tau_{dyn}} \sim \frac{m}{Rr^2\rho} = \frac{r}{R}\frac{\langle\rho_{\rm c}\rangle}{\rho}. \tag{3}$$

As an example, consider the engulfment of a planet by a star with a radiative envelope. In the outermost layers of the star, the density is very low, $\rho \ll \langle\rho_c\rangle$, so the inspiral timescale is very long – the orbit is almost circular. However, the density rapidly increases inward. Since $r \ll R$, long before the tidal disruption of the planet at $\langle\rho_c\rangle \sim \rho$, the inspiral timescale drops below the dynamical timescale once $\rho \sim \langle\rho_c\rangle(r/R)$. At this point, the orbital motion stalls, and the planet transitions to a largely radial infall with a terminal velocity determined by equating the gravitational acceleration to drag, $GM/R^2 \sim r^2 v_{term}^2 \rho$. (Of course, the planet may ablate even sooner: the typical energy release by the time the planet moves to radius $x \ll R$ is $\Delta E \sim GMm/x$, and if this energy is primarily deposited into the planet rather than the star, it can ablate once the deposited energy exceeds the planet's binding energy, $\Delta E > Gm^2/r$.)

Note that with the drag and accretion coefficients included,

$$\frac{dE}{E} = -\frac{C_D}{C_A}\frac{dm}{m}, \tag{4}$$

so $E_0/E = (m/m_0)^{C_D/C_A}$. E.g., for $C_D = C_A = 1$, the energy decay timescale is the same as the mass growth timescale. If we believe that neutron stars can spiral in by several orders of magnitude within a common envelope while barely changing their mass, it must be the case that $C_A \ll C_D$.

It's worth briefly commenting on our assumption of supersonic motion. The sound speed is approximately $c_s^2 \sim P/\rho$. Because the donor is in hydrostatic equilibrium, $dP/dr = GM\rho/R^2$. Thus, on average, equating $dP/dr \sim P/R$, we would conclude that $c_s^2 \sim P/\rho \sim GM/R \sim v^2$ – i.e., the sound speed is of the order of the Keplerian orbital velocity and our calculation is roughly accurate. More precisely, the validity of this assumption depends on the details of the envelope structure.